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Functions with bounded variation on a class of Riemannian manifolds with Ricci curvature unbounded from below

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Abstract

After establishing some new global facts (like a measure theoretic structure theorem and a new invariant characterization of Sobolev functions) about complex-valued functions with bounded variation on arbitrary noncompact Riemannian manifolds, we extend results of Miranda/the second author/Paronetto/Preunkert and of Carbonaro/Mauceri on the heat semigroup characterization of the variation of L^1-functions to a class of Riemannian manifolds with possibly unbounded from below Ricci curvature.

1 Introduction

In the last decades, the class of integrable functions with bounded variation (BV) has proven to be very well suited for the formulation of geometric variation problems in the *Euclidean* $\mathbb{R}^m$. The essential advantage of BV in this context when compared with the smaller Sobolev class $\mathsf{W}^{1,1}$ is the possibility of allowing discontinuities along hypersurfaces. For example, the class of characteristic functions of Caccioppoli sets is a subclass of BV which turns out to be the appropriate class to formulate the isoperimetric problem. We refer the reader to [2] for various aspects of BV functions in Euclidean space.

The definition of the variation of a function is intuitively clear and classical

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for the case of one variable, $m = 1$, while the first satisfactory (from the geometric/variational point of view) extension of this definition to arbitrary m which descends to the usual definition if $m = 1$ has been given in [12], where De Giorgi defines the variation of $f \in \mathsf{L}^1(\mathbb{R}^m, \mathbb{R})$ to be the well-defined quantity

$$\lim_{t \to 0+} \int_{\mathbb{R}^m} |\mathrm{grad}(\mathrm{e}^{t\Delta} f)|(x) \mathrm{d}x \in [0, \infty].$$

The main result of [12] implies

$$\begin{aligned} \mathrm{Var}(f) &:= \lim_{t \to 0+} \int_{\mathbb{R}^m} |\mathrm{grad}(\mathrm{e}^{t\Delta} f)|(x) \mathrm{d}x \qquad (1) \\ &= \sup \left\{ \int_{\mathbb{R}^m} f(x) \mathrm{div} \alpha(x) \mathrm{d}x \,\middle|\, \alpha \in [\mathsf{C}_0^\infty(\mathbb{R}^m, \mathbb{R})]^m, \|\alpha\|_\infty \leq 1 \right\}, \end{aligned}$$

an equality which turned out to be the starting point for all the above mentioned Euclidean results. Indeed, it follows from (1) that $\mathrm{Var}(f)$ is finite, if and only if the distributional gradient of f is given by integrating against a $\mathbb{R}^m$-valued Borel measure.
Noting that all of the data in (1) can be defined invariantly using differential forms and the exterior derivative (cf. Definition 2.2 below) and in view of the importance of (1) for the Euclidean case, the aim of this paper is to formulate and prove the above result in a very large class of noncompact Riemannian manifolds, where we want to allow complex-valued functions.
Our main result here is Theorem 3.3, which essentially reads as follows:

Let M be a geodesically complete smooth Riemannian manifold whose Ricci curvature $\mathscr{R}$ admits a decomposition $\mathscr{R} = \mathscr{R}_1 - \mathscr{R}_2$ into pointwise self-adjoint Borel sections $\mathscr{R}_1, \mathscr{R}_2 \geq 0$ in $\mathrm{End}(\mathrm{T}^*M)$ such that $|\mathscr{R}_2| \in \mathcal{K}(M)$, the Kato class of M. Then for any $f \in \mathsf{L}^1(M)$, the complex-valued version of the equality in (1) holds true.

As bounded functions are always in the Kato class (see section 3), the above result extends the results of [28, 10] in the sense that we do not have to assume that $\mathscr{R}$ is bounded from below, the latter condition being equivalent to $|\mathscr{R}_-| \in \mathsf{L}^\infty(M)$, where $\mathscr{R}_-$ is the negative part of the Ricci curvature ($\mathscr{R}_-$ can be defined using the spectral calculus on the fibers of T^*M). In fact, we can drop the latter condition using probabilistic techniques: it follows from Weitzenböck's formula that the heat semigroup given by the Laplace operator on 1-forms is a *generalized Schrödinger semigroup* in the sense of [20] whose potential term is given by $\mathscr{R}$. Thus we can combine the results from [20] on probabilistic formulae for such semigroups together with a new probabilistic estimate on Kato functions (see Lemma 3.8 below), which turns out to be just enough to derive the equality in (1). As a consequence of Theorem 3.3

we can derive L^p-type criteria on the Ricci curvature that imply the validity of (1). Furthermore, it is also possible to derive a certain stability result of (1) under a class of conformal transformations of the underlying Riemannian structure.
Let us remark that the Kato condition has been used for decades in order to deal with *local* singularities of potentials in quantum mechanics [24, 1, 32, 8, 34, 21]. On the other side, as far as we know, this paper is the first one where the Kato condition is used in order to deal with possibly *globally* unbounded potential terms in a purely Riemann geometric setting.

This paper is organized as follows:

Defining the variation by the right hand[1] side of the invariant version of (1), we first collect several properties of the variation that are valid for arbitrary (geodesically complete) Riemannian manifolds in section 2. For example, we prove that functions having bounded variation are precisely those whose distributional derivative is a generalized vector measure, a new structure theorem, which produces the classical structure theorem in the Euclidean case, but can become technical for nonparallelizable manifolds (see Theorem 2.5). This structure theorem produces also a new global characterization of first order Sobolev functions (see Proposition 2.11). Among other results, we have also added further equivalent characterizations of the variation to the latter section, like an approximation result (see Theorem 2.12), and also an invariance result of BV under quasi-isometric changes of the Riemannian structure (see Corollary 2.13).
Section 3 is devoted to the formulation and the proof of our main result Theorem 3.3, as well as the before mentioned L^p-type criteria for the validity of (1) (see Corollary 3.5), and the stability result of the latter equality under certain conformal transformations (see Corollary 3.7). In section 3, we also recall the necessary definitions and facts about Brownian motion and the Kato class $\mathcal{K}(M)$ of M.
Finally, we have collected some abstract facts on vector measures on locally compact spaces in an appendix.

2 Setting and general facts on the variation

Let M denote a smooth connected Riemannian manifold without boundary, with $\mathrm{vol}(\mathrm{d}x)$ the Riemannian volume measure, and $\mathrm{K}_r(x)$ the open geodesic ball with radius r around x. Unless otherwise stated, the underlying fixed

[1] We prefer the right hand side instead of the left hand side, as the former also makes for *locally* integrable functions

Riemannian structure will be ommitted in the notation. We set $m := \dim M$. If $E \to M$ is a smooth Hermitian vector bundle, then, abusing the notation in the usual way, the Hermitian structure will be denoted with $(\bullet, \bullet)_x$, $x \in M$; moreover, $|\bullet|_x$ will stand for the norm and the operator norm corresponding to $(\bullet, \bullet)_x$ on each fiber E_x, and $\langle \bullet, \bullet \rangle$ for the inner product in the Hilbert space $\Gamma_{\mathsf{L}^2}(M, E)$, that is,

$$\langle f_1, f_2 \rangle = \int_M (f_1(x), f_2(x))_x \mathrm{vol}(\mathrm{d}x). \tag{2}$$

Furthermore, the norm $\|\bullet\|_p$ on $\Gamma_{\mathsf{L}^p}(M, E)$ is given by

$$\|f\|_p = \left(\int_M |f(x)|_x^p \mathrm{vol}(\mathrm{d}x) \right)^{1/p} \tag{3}$$

if $p \in [1, \infty)$, and $\|f\|_\infty$ is given by the infimum of all $C \geq 0$ such that $|f(x)|_x \leq C$ for a.e. $x \in M$. The corresponding operator norms on the spaces of bounded linear operators $\mathscr{L}(\Gamma_{\mathsf{L}^p}(M, E))$ will also be denoted with $\|\bullet\|_p$. If $\tilde{E} \to M$ is a second bundle as above and if

$$D \colon \Gamma_{\mathsf{C}^\infty}(M, E) \longrightarrow \Gamma_{\mathsf{C}^\infty}(M, \tilde{E})$$

is a linear differential operator, then we denote with $D^\dagger$ the formal adjoint of D with respect to (2).

We will apply the above in the following situation: For any $k = 0, \dots, m$ we will consider the smooth Hermitian[2] vector bundle of k-forms $\bigwedge^k \mathrm{T}^* M \to M$, with

$$\Omega^k_{\mathscr{C}}(M) := \Gamma_{\mathscr{C}}\Big(M, \bigwedge^k \mathrm{T}^* M\Big), \text{ where } \mathscr{C} = \mathsf{C}^\infty, \mathsf{L}^p, \text{etc.}$$

In order to make the notation consistent, we will set $\bigwedge^0 \mathrm{T}^* M := M \times \mathbb{C}$, where of course $\mathscr{C}(M) = \Omega^0_{\mathscr{C}}(M)$ for functions. In particular, all function spaces are spaces of *complex-valued* functions. The subscript "0" in $\mathscr{C}_0$ will always stand for "compactly supported elements of $\mathscr{C}$". Whenever necessary, we will write $\mathscr{C}_{\mathbb{R}}$ for the real-valued elements of $\mathscr{C}$. If

$$\mathrm{d}_k : \Omega^k_{\mathsf{C}^\infty}(M) \longrightarrow \Omega^{k+1}_{\mathsf{C}^\infty}(M)$$

stands for the exterior derivative, then the Laplace-Beltrami operator acting on k-forms on M is given as

$$-\Delta_k := \mathrm{d}_k^\dagger \mathrm{d}_k + \mathrm{d}_{k-1} \mathrm{d}_{k-1}^\dagger : \Omega^k_{\mathsf{C}^\infty}(M) \longrightarrow \Omega^k_{\mathsf{C}^\infty}(M).$$

[2] so everything has been complexified

We shall write $\mathrm{d} := \mathrm{d}_0$ for the exterior derivative on functions, so that $-\Delta := -\Delta_0 = \mathrm{d}^\dagger \mathrm{d}$.

To make contact with the introduction, we add:

Remark 2.1. If $\alpha \in \Omega^1_{\mathsf{C}^\infty}(M)$ and if $X^{(\alpha)}$ is the smooth vector field on M corresponding to α by the Riemannian duality on M, then an integration by parts shows $\mathrm{d}^\dagger \alpha = -\mathrm{div} X^{(\alpha)}$. Here, the divergence $\mathrm{div}(X) \in \mathsf{C}^\infty(M)$ of a smooth vector field X on M is defined in the usual way as follows: locally, if

$$X = \sum_{j=1}^m X^j \frac{\partial}{\partial x^j} \text{ with } X^j \in \mathsf{C}^\infty(M), \text{ then } \mathrm{div} X = \sum_{j=1}^m \frac{\partial}{\partial x^j} X^j.$$

Furthermore, if $f \in \mathsf{C}^\infty(M)$, then $\mathrm{grad} f$ is the smooth vector field on M given by $\mathrm{grad} f = X^{(\mathrm{d}f)}$, where locally

$$\mathrm{d}f = \sum_{j=1}^m \frac{\partial f}{\partial x^j} \mathrm{d}x^j.$$

The Friedrichs realization of $-\Delta_k/2$ in $\Omega^k_{\mathsf{L}^2}(M)$ will be denoted with $H_k \geq 0$, where again $H := H_0$ on functions. We will freely use the fact that for any $p \in [1, \infty]$ the strongly continuous self-adjoint semigroup of contractions $(\mathrm{e}^{-tH})_{t \geq 0} \subset \mathscr{L}(\mathsf{L}^2(M))$ uniquely extends to a strongly continuous semi-group of contractions $(\mathrm{e}^{-tH})_{t \geq 0} \subset \mathscr{L}(\mathsf{L}^p(M))$, and by local elliptic regularity, $\mathrm{e}^{-tH} f$ has a smooth representative $\mathrm{e}^{-tH} f(\bullet)$ for any $t > 0$, $f \in \mathsf{L}^p(M)$.

The following definition is a generalization of Definition (1.4) in [28] to *complex-valued* and *locally* integrable functions:

Definition 2.2. Let $f \in \mathsf{L}^1_{\mathrm{loc}}(M)$. Then the quantity

$$\begin{aligned} \mathrm{Var}(f) :&= \sup\left\{ \left| \int_M \overline{f(x)} \mathrm{d}^\dagger \alpha(x) \mathrm{vol}(\mathrm{d}x) \right| \,\middle|\, \alpha \in \Omega^1_{\mathsf{C}^\infty_0}(M), \|\alpha\|_\infty \leq 1 \right\} \\ &\in [0, \infty] \end{aligned} \tag{4}$$

is called the *variation* of f, and f is said to have *bounded variation* if $\mathrm{Var}(f) < \infty$.

Note the following trivial equalities for any $f \in \mathsf{L}^1_{\mathrm{loc}}(M)$,

$$\begin{aligned} \mathrm{Var}(f) &= \sup\left\{ \left| \int_M f(x) \overline{\mathrm{d}^\dagger \alpha(x)} \mathrm{vol}(\mathrm{d}x) \right| \,\middle|\, \alpha \in \Omega^1_{\mathsf{C}^\infty_0}(M), \|\alpha\|_\infty \leq 1 \right\} \\ &= \sup\left\{ \left| \int_M f(x) \mathrm{d}^\dagger \alpha(x) \mathrm{vol}(\mathrm{d}x) \right| \,\middle|\, \alpha \in \Omega^1_{\mathsf{C}^\infty_0}(M), \|\alpha\|_\infty \leq 1 \right\}, \end{aligned}$$

and that of course the property $\mathrm{Var}(f) < \infty$ depends very sensitively on the Riemannian structure of M (see Corollary 2.13 below for a certain stability). We will now collect some properties of the variation which are valid without any assumptions on the Riemannian structure of M.

As in the Euclidean case, the variation of a *smooth* function f can be calculated explicitly from the L^1-norm of $\mathrm{d}f$:

Proposition 2.3. *For all $f \in \mathsf{C}^\infty(M)$ one has* $\mathrm{Var}(f) = \|\mathrm{d}f\|_1$.

Proof. Here, $\mathrm{Var}(f) \leq \cdots$ is clear from

$$\int_M \overline{f(x)} \mathrm{d}^\dagger \alpha(x) \mathrm{vol}(\mathrm{d}x) = \int_M (\mathrm{d}f(x), \alpha(x))_x \mathrm{vol}(\mathrm{d}x) \quad \text{for any } \alpha \in \Omega^1_{\mathsf{C}_0^\infty}(M).$$

In order to prove $\mathrm{Var}(f) \geq \cdots$, note that $\{\mathrm{d}f \neq 0\} \subset M$ is an open subset, so that there is a sequence $(\psi_n) \subset \mathsf{C}_0^\infty(\{\mathrm{d}f \neq 0\})$ such that $0 \leq \psi_n \leq 1$ and $\psi_n \to 1$ as $n \to \infty$ pointwise. Then

$$\alpha_n := \frac{\psi_n}{|\mathrm{d}f|} \mathrm{d}f \in \Omega^1_{\mathsf{C}_0^\infty}(\{\mathrm{d}f \neq 0\}) \subset \Omega^1_{\mathsf{C}_0^\infty}(M),$$

one has $(\mathrm{d}f, \alpha_n) \geq 0$, $\|\alpha_n\|_\infty \leq 1$, and we get

$$\begin{aligned} \int_M |\mathrm{d}f(x)|_x \mathrm{vol}(\mathrm{d}x) &\leq \liminf_{n\to\infty} \int_M (\mathrm{d}f(x), \alpha_n(x))_x \mathrm{vol}(\mathrm{d}x) \\ &= \liminf_{n\to\infty} \int_M \overline{f(x)} \mathrm{d}^\dagger \alpha_n(x) \mathrm{vol}(\mathrm{d}x) \leq \mathrm{Var}(f), \end{aligned}$$

where we have used Fatou's lemma. ∎

We continue with the distributional properties of the differential of bounded variation functions: In the m-dimensional Euclidean situation, it is known (see for example Corollary 2.6 below) that there is a bijection between functions with bounded variation and the Banach space of classical $\mathbb{C}^m$-valued Borel measures. On a nonparallelizable manifold this need not be the case anymore. However, we found a *global* statement, which is formulated as Theorem 2.5 below, and which locally produces precisely the above statements (see also Proposition A.1). The essential idea is to use a Banach space of "generalized vector measures on M".
To this end, let C_∞ stand for the class of sections in Hermitian vector bundles that vanish at infinity. Clearly, $\Omega^1_{\mathsf{C}_\infty}(M)$ becomes a complex Banach space with respect to $\|\bullet\|_\infty$.

Definition 2.4. The Banach dual $(\Omega^1_{\mathsf{C}_\infty}(M))^*$ is called the *space of generalized vector measures on M.*

We denote the canonical norm on the space of generalized vector measures with $\|\bullet\|_{\infty,*}$.
Using Friedrichs-mollifiers and a standard partition of unity argument one finds that the elements of $\mathsf{C}_0(M)$ can be approximated in $\|\bullet\|_\infty$ by $\mathsf{C}_0^\infty(M)$. Using now that $\mathsf{C}_0(M)$ is dense $\mathsf{C}_\infty(M)$, together with a second localization argument, one gets that $\Omega^1_{\mathsf{C}_0^\infty}(M)$ is dense in $\Omega^1_{\mathsf{C}_\infty}(M)$. Thus whenever a linear functional T on $\Omega^1_{\mathsf{C}_0^\infty}(M)$ satisfies

$$\|T\|_{\infty,*} := \sup\Big\{ |T(\alpha)| \Big| \alpha \in \Omega^1_{\mathsf{C}_0^\infty}(M), \|\alpha\|_\infty \leq 1\Big\} < \infty,$$

it can be uniquely extended to an element of $(\Omega^1_{\mathsf{C}_\infty}(M))^*$ with the same norm. Let us furthermore denote with $\mathscr{M}(M)$ the space of equivalence classes $[(\mu,\sigma)]$ of pairs (μ,σ) with μ a finite positive Borel measure on M and σ a Borel section in T^*M with $|\sigma| = 1$ μ-a.e. in M, where

$$\begin{aligned}(\mu,\sigma) \sim (\mu',\sigma') :\Leftrightarrow \mu = \mu' \text{ as Borel measures}\\ \text{and } \sigma(x) = \sigma'(x) \text{ for } \mu/\mu' \text{ a.e. } x \in M.\end{aligned}$$

Now we can formulate the following structure theorem, which in particular gives a global justification of Definition 2.4:

Theorem 2.5. a) *The map*

$$\Psi : \mathscr{M}(M) \longrightarrow (\Omega^1_{\mathsf{C}_\infty}(M))^*, \quad \Psi[(\mu,\sigma)](\alpha) := \int_M (\sigma,\alpha)\mathrm{d}\mu$$

is a well-defined bijection with $\|\Psi[(\mu,\sigma)]\|_{\infty,*} = \mu(M)$.

b) *A function* $f \in \mathsf{L}^1_{\mathrm{loc}}(M)$ *has bounded variation if and only if* $\|\mathrm{d}f\|_{\infty,*} < \infty$*, and then one has* $\mathrm{Var}(f) = \|\mathrm{d}f\|_{\infty,*}$.

Proof. a) Clearly, Ψ is a well-defined map. We divide the proof into three parts:

1.Ψ *is surjective*: Let T be a generalized vector measure and consider the functional given by

$$\tilde{\tilde{T}}(f) := \sup\Big\{ |T(f\alpha)| \Big| \alpha \in \Omega^1_{\mathsf{C}_\infty}(M), \|\alpha\|_\infty \leq 1\Big\}, \ 0 \leq f \in \mathsf{C}_\infty(M). \quad (5)$$

For every test form $\alpha \in \Omega^1_{\mathsf{C}_\infty}(M)$ and every $0 \leq f \in \mathsf{C}_\infty(M)$ with $T(f\alpha) \neq 0$ set

$$z = \frac{\overline{T(f\alpha)}}{|T(f\alpha)|} \in \mathbb{C},$$

whence

$$|T(f\alpha)| = zT(f\alpha) = T(zf\alpha) = \mathrm{Re}\Big(T(fz\alpha)\Big).$$

Since $|z| = 1$, $z\alpha$ is again an admissible test form as well, and we get

$$|T(f\alpha)| \le \sup\left\{ \mathrm{Re}\Big(T(f\omega)\Big) \,\Big|\, \omega \in \Omega^1_{\mathsf{C}_\infty}(M),\ \|\omega\|_\infty \le 1 \right\},$$

so that

$$\tilde{\tilde{T}}(f) = \sup\left\{ \mathrm{Re}\Big(T(f\alpha)\Big) \,\Big|\, \alpha \in \Omega^1_{\mathsf{C}_\infty}(M),\ \|\alpha\|_\infty \le 1 \right\}.$$

Thus from demposing the real- and imaginary parts of functions into their positive and negative parts, it follows that $\tilde{\tilde{T}}$ has a unique extension to a positive bounded linear functional $\tilde{T}$ on $\mathsf{C}_\infty(X)$, and by Riesz-Markoff's theorem (see Proposition A.1 a)), there is a unique finite positive Borel measure μ on M such that $\int_M f \mathrm{d}\mu = \tilde{T}(f)$ for all $f \in \mathsf{C}_\infty(M)$. Furthermore, μ satisfies

$$\mu(M) = \sup\left\{ |\tilde{T}(f)| \,\Big|\, f \in \mathsf{C}_\infty(M), \|f\|_\infty \le 1 \right\} < \infty.$$

By the paracompactness of M there is a finite open cover $\bigcup_{l=1}^d U_l = M$ (with U_l possibly disconnected and with noncompact closure) such that for any l there is an orthonormal basis $e_1^{(l)}, \dots, e_m^{(l)} \in \Omega^1_{\mathsf{C}^\infty}(U_l)$. We also take a partition of unity (ψ_l) subordinate to (U_l), that is, $\psi_l \in \mathsf{C}^\infty(M)$, $0 \le \psi_l \le 1$ and $\sum_l \psi_l = 1$ pointwise, and $\mathrm{supp}(\psi_l) \subset U_l$. Then by the above considerations, the assignment $f \mapsto T(f\psi_l e_j^{(l)})$, $f \in \mathsf{C}_\infty(U_l)$, extends to a bounded linear functional on $\mathsf{L}^1(U_l, \mu)$ with norm $\le \mu(M)$, so that there is a $\sigma_j^{(l)} \in \mathsf{L}^\infty(U_l, \mu)$ with $|\sigma_j^{(l)}| \le 1$ μ-a.e. such that

$$T(fe_j^{(l)}) = \int_{U_l} \sigma_j^{(l)}(x) f(x) \mu(\mathrm{d}x) \ \text{ for any } f \in \mathsf{C}_\infty(U_l).$$

Now it is easily checked that the Borel 1-form $\sigma := \sum_{l=1}^d \psi_l \sum_{j=1}^m \sigma_j^{(l)} e_j^{(l)}$ on M satisfies $|\sigma| \le 1$ μ-a.e. and $T = \int_M (\sigma, \bullet) \mathrm{d}\mu$. Finally, since

$$\begin{aligned}
&\int_M |\sigma(x)|_x \mu(\mathrm{d}x) \\
&\ge \sup\left\{ \left| \int_M (\sigma, \alpha) \mathrm{d}\mu \right| \,\Big|\, \alpha \in \Omega^1_{\mathsf{C}_\infty}(M),\ \|\alpha\|_\infty \le 1 \right\} \\
&= \sup\left\{ |\tilde{T}(f)| \,\Big|\, f \in \mathsf{C}_\infty(M), \|f\|_\infty \le 1 \right\} \\
&= \mu(M),
\end{aligned}$$

we get $|\sigma| = 1$ μ-a.e., which completes the proof of the asserted surjectivity of Ψ.

2.Ψ *is injective*: If $\Psi[(\mu,\sigma)](\alpha) = \Psi[(\mu',\sigma')](\alpha)$ for all $\alpha \in \Omega^1_{\mathsf{C}_\infty}(M)$, then one has $\mu = \mu'$ as Borel measures: Indeed, using the above notation, for every Borel set $N \subset M$ we have

$$\begin{aligned}
&\mu(N)\\
&= \sup\left\{ \widetilde{\widetilde{\Psi[(\mu,\sigma)]}}(f) \;\middle|\; f \in \mathsf{C}_\infty(M), 0 \le f \le 1_N \right\}\\
&= \sup\left\{ |\Psi[(\mu,\sigma)](f\alpha)| \;\middle|\; f \in \mathsf{C}_\infty(M), \alpha \in \Omega^1_{\mathsf{C}_\infty}(M), |\alpha| \le 1, 0 \le f \le 1_N \right\}\\
&= \sup\left\{ |\Psi[(\mu,\sigma)](\beta)| \;\middle|\; \beta \in \Omega^1_{\mathsf{C}_\infty}(M), |\beta| \le 1_N \right\}\\
&= \sup\left\{ \left|\int_M (\sigma',\beta)\mathrm{d}\mu'\right| \;\middle|\; \beta \in \Omega^1_{\mathsf{C}_\infty}(M), |\beta| \le 1_N \right\}\\
&\le \mu'(N). \qquad (6)
\end{aligned}$$

Exchanging μ with μ' we get $\mu(N) = \mu'(N)$ for all Borel sets $N \subset M$, as claimed. To see that $\sigma = \sigma'$ μ/μ'-a.e., let us observe that $T := \Psi[(\mu,\sigma)]$ extends to $\Omega^1_{\mathsf{B}_b}(M)$, where B_b denotes the space of bounded Borel functions, as for every $\alpha \in \Omega^1_{\mathsf{B}_b}(M)$ the function $x \mapsto (\alpha(x),\sigma(x))_x$ on M belongs to $\mathsf{L}^1(M,\mu)$ and we may define $T(\alpha) = \int_M (\alpha,\sigma)\mathrm{d}\mu$. Therefore, we may apply T to σ and use that $T(\sigma) = \Psi[(\mu',\sigma')](\sigma)$ and the equality $\mu = \mu'$ to get

$$\mu(M) = T(\sigma) = \int_M (\sigma,\sigma')\mathrm{d}\mu' = \int_M (\sigma,\sigma')\mathrm{d}\mu \le \int_M |(\sigma,\sigma')|\mathrm{d}\mu$$

and the equality $|(\sigma,\sigma')| = 1$ μ-a.e follows. Since $|\sigma| \le 1$ and $|\sigma'| \le 1$ we deduce $\sigma = \sigma'$ μ-a.e.

3. *One has* $\|\Psi[(\mu,\sigma)]\|_{\infty,*} = \mu(M)$: Indeed, this follows from the bijectivity of Ψ and the proof of the surjectivity.

b) If $\|\mathrm{d}f\|_{\infty,*} < \infty$, then clearly one has $\mathrm{Var}(f) \le \left\|\tilde{\mathrm{d}f}\right\|_{\infty,*}$ by the very definition of $\mathrm{d}f$, namely,

$$\mathrm{d}f(\alpha) = \int_M \overline{f(x)}\mathrm{d}^\dagger\alpha(x)\mathrm{vol}(\mathrm{d}x) \text{ for any } \alpha \in \Omega^1_{\mathsf{C}_0^\infty}(M).$$

Conversely, if $\mathrm{Var}(f)$ is finite, then by a homogeneity argument, one has the following estimate

$$\left|\int_M \overline{f(x)}\mathrm{d}^\dagger\alpha(x)\mathrm{vol}(\mathrm{d}x)\right| \le \mathrm{Var}(f)\|\alpha\|_\infty \text{ for any } \alpha \in \Omega^1_{\mathsf{C}_0^\infty}(M),$$

which implies $\|\mathrm{d}f\|_{\infty,*} \le \mathrm{Var}(f) < \infty$. ■

If f has bounded variation we will write $|\mathrm{D}f|$ for the measure, and σ_f for the $|\mathrm{D}f|$-equivalence class of sections corresponding to $\Psi^{-1}(\mathrm{d}f)$, so that we have

$$\mathrm{d}f(\alpha) = \int_M (\sigma_f(x), \alpha(x))_x |\mathrm{D}f|(\mathrm{d}x) \quad \text{for any } \alpha \in \Omega^1_{\mathsf{C}_0^\infty}(M). \tag{7}$$

We directly recover the following complex variant of a classical result, which in particular states that locally, complex-valued bounded variation functions can be considered as $\mathbb{R}^2$-valued bounded variation functions and vice versa:

Corollary 2.6. *Let $M = U$, with $U \subset \mathbb{R}^m$ a domain with its Euclidean metric, and let $f \in \mathsf{L}^1_{\mathrm{loc}}(U)$. Then one has*

$$\begin{aligned}&\mathrm{Var}(f) = \\ &\sup\left\{ \int_U \Big(\mathrm{Re}(f)\mathrm{div}\alpha_1 + \mathrm{Im}(f)\mathrm{div}\alpha_2\Big)\,|_x \,\mathrm{d}x \,\middle|\, \alpha \in [\mathsf{C}^\infty_{0,\mathbb{R}}(U)]^{2m}, \|\alpha\|_\infty \leq 1 \right\},\end{aligned} \tag{8}$$

where in the above set any $\alpha \in [\mathsf{C}^\infty_{0,\mathbb{R}}(U)]^{2m}$ is written as $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_j \in [\mathsf{C}^\infty_{0,\mathbb{R}}(U)]^m$. Furthermore, f has bounded variation if and only if there is a (necessarily unique) $\mathbb{C}^m$-valued Borel measure $\mathrm{D}f$ on U such that[3] $\mathrm{grad} f = \mathrm{D}f$ *as distributions, and then it holds that* $\mathrm{Var}(f) = |\mathrm{D}f|(U)$.

Proof. If f has bounded variation, then with standard identifications, Theorem 2.5 implies the existence of $\mathrm{D}f$ in a way that $\mathrm{Var}(f) = |\mathrm{D}f|(U)$, and Proposition A.1 b) implies $|\mathrm{D}f|(U) = |(\mathrm{D}f)_{\mathbb{R}^{2m}}|(U)$, where $|(\mathrm{D}f)_{\mathbb{R}^{2m}}|(U)$ is well-known to be equal to the supremum in (8) (see for example the proof of Proposition 3.6 in [2]). If f has infinite variation, then we can conclude analogously. ■

Remark 2.7. Note that in the situation of Corollary 2.6, the equality of $\mathrm{Var}(f)$ to the supremum in (8) is not immediate from Definition 2.2, where a complex absolute value appears. But it is precisely this equality that makes our definition of variation the natural one in the case of complex-valued functions on the most fundamental level, which is the case of functions of one variable: Indeed, let $I \subset \mathbb{R}$ be an open interval. Then the characterization of the variation by the supremum in (8) combined with Theorem 3.27 in [2] (this is a highly nontrivial fact) implies that for any $f \in \mathsf{L}^1_{\mathrm{loc}}(I)$ one has

$$\mathrm{Var}(f) = \inf_{f(\bullet) \in f} \sup\left\{ \sum_{j=1}^{n-1} |f(x_{j+1}) - f(x_j)| \,\middle|\, n \geq 2,\, x_1 < x_2 \cdots < x_n \right\}. \tag{9}$$

[3]Here, $|\mathrm{D}f|$ stands for the total variation measure corresponding to the vector measure $\mathrm{D}f$ (cf. Section A). Of course this notation is consistent with (7).

Note here that f is an equivalence class, so the infimum in (9) is taken among all functions coinciding with f a.e. in I. Indeed, $\sup\{\dots\}$ depends heavily on the particular representative of $f(\bullet) : I \to \mathbb{C}$ of f.

We continue with our general observations. Let $p \in [1,\infty)$ and recall that a countable system of seminorms on $\mathsf{L}^p_{\mathrm{loc}}(M)$ is given through $f \mapsto \int_{K_n} |f|^p \mathrm{d}\mu$, where (K_n) is an exhaustion of M with relatively compact domains and μ is a smooth positive Borel measure on M, that is, the restriction of μ to an arbitrary chart has a positive smooth density function with respect to the m-dimensional Lebesgue measure. Then the corresponding locally convex topology does not depend on the particular choice of (K_n) and μ, and furthermore for any fixed $\psi \in \mathsf{C}^\infty_0(M)$, the map

$$\mathsf{L}^1_{\mathrm{loc}}(M) \longrightarrow [0,\infty),\ f \longmapsto \left| \int_M \overline{f(x)}\psi(x)\mathrm{vol}(\mathrm{d}x) \right|$$

is continuous. This observation directly implies:

Proposition 2.8. *For any $p \in [1,\infty)$ the maps*

$$\begin{aligned} \mathsf{L}^p_{\mathrm{loc}}(M) &\longrightarrow [0,\infty],\ \ f \longmapsto \mathrm{Var}(f) \\ \mathsf{L}^p(M) &\longrightarrow [0,\infty],\ \ f \longmapsto \mathrm{Var}(f) \end{aligned}$$

are lower semicontinuous with respect to the corresponding canonical topologies.

We also have the following fact, which follows easily from the completeness of $\mathsf{L}^1(M)$ and Proposition 2.8:

Proposition 2.9. *The space*

$$\mathsf{BV}(M) := \left\{ f \middle| \, f \in \mathsf{L}^1(M), \mathrm{Var}(f) < \infty \right\}$$

is a complex Banach space with respect to the norm $\|f\|_{\mathsf{BV}} := \|f\|_1 + \mathrm{Var}(f)$.

Next we shall record that first-order L^1-Sobolev functions belong to $\mathsf{BV}(M)$ with the same norm. To this end, for any $p \in [1,\infty)$ we denote the complex Banach space of first order L^p-Sobolev functions with

$$\mathsf{W}^{1,p}(M) := \left\{ f \middle| \, f \in \mathsf{L}^p(M), \mathrm{d}f \in \Omega^1_{\mathsf{L}^p}(M) \right\}$$

with its canonical norm $\|f\|_{1,p} := \|f\|_p + \|\mathrm{d}f\|_p$. Furthermore, we define $\mathsf{H}^{1,p}(M) \subset \mathsf{W}^{1,p}(M)$ as the closure of the space of functions $f \in \mathsf{L}^p(M) \cap$

$\mathsf{C}^\infty(M)$ such that $\mathrm{d}f \in \Omega^1_{\mathsf{L}^p}(M)$ with respect to $\|\bullet\|_{1,p}$. The equality $\mathsf{W}^{1,p} = \mathsf{H}^{1,p}$ is known to hold on open subsets of the Eucliden $\mathbb{R}^m$ by Meyers-Serrin's Theorem [26]. It seems to be unknown whether this extends to abitrary M. However, one has the following result under geodesic completenes, which should be known, but which we have not been able to find a direct reference for (note here that Theorem 1 in [4] only states that $\mathsf{C}^\infty_0(M)$ is dense in $\mathsf{H}^{1,p}(M)$). It relies on the existence of first order cut-off functions:

Proposition 2.10. *If M is geodesically complete, then $\mathsf{C}^\infty_0(M)$ is dense in $\mathsf{W}^{1,p}(M)$, in particular, one has $\mathsf{W}^{1,p}(M) = \mathsf{H}^{1,p}(M)$.*

Proof. Under geodesic completeness, there is a sequence of functions $(\psi_n) \subset \mathsf{C}^\infty_0(M)$ with $0 \leq \psi_n \leq 1$, $\psi_n \to 1$ pointwise and $\|\mathrm{d}\psi_n\|_\infty \to 0$ as $n \to \infty$ (see [31], Proposition 4.1). For $f \in \mathsf{W}^{1,p}(M)$ let $f_n := \psi_n f$. Then the Sobolev product rule

$$\mathrm{d}f_n = f\mathrm{d}\psi_n + \psi_n \mathrm{d}f$$

implies that $f_n \in \mathsf{W}^{1,p}_0(M)$ (the compactly supported elements in $\mathsf{W}^{1,p}(M)$!) and also that $f_n \to f$ in $\|\bullet\|_{1,p}$, the latter from dominated convergence. Thus $\mathsf{W}^{1,p}_0(M)$ is dense in $\mathsf{W}^{1,p}(M)$ and it remains to show that functions from the former space can be approximated by functions in $\mathsf{C}^\infty_0(M)$. However, now one can use a partition of unity argument corresponding to a *finite* atlas for M to see that it is sufficient to prove that for an open subset U of the Euclidean $\mathbb{R}^m$, the space $\mathsf{C}^\infty_0(U)$ is dense in the normed space $\mathsf{W}^{1,p}_0(U)$. The latter fact is well-known. ■

The following proposition completely clarifies the connection between Sobolev- and BV-functions:

Proposition 2.11. a) *One has $\|f\|_{\mathsf{BV}} = \|f\|_{1,1}$ for all $f \in \mathsf{W}^{1,1}(M)$. In particular, $\mathsf{H}^{1,1}(M)$ and $\mathsf{W}^{1,1}(M)$ are closed subspaces of $\mathsf{BV}(M)$.*

b) *Any $f \in \mathsf{BV}(M)$ is in $\mathsf{W}^{1,1}(M)$, if and only if with the notation from* (7) *it holds that $|\mathrm{D}f| \ll \mathrm{vol}$ as Borel measures.*

Proof. a) As one has

$$\mathrm{d}f(\alpha) = \int_M (\sigma(x), \alpha(x))_x \mu(\mathrm{d}x) \text{ for any } \alpha \in \Omega^1_{\mathsf{C}^\infty_0}(M),$$

where $\sigma := \mathrm{d}f/|\mathrm{d}f|$ and $\mu := |\mathrm{d}f|\mathrm{vol}$, the claim follows immediately from Theorem 2.5.

b) In view of (7), if for some $0 \leq \rho \in \mathsf{L}^1(M)$ one has $|\mathrm{D}f| = \rho\,\mathrm{vol}$, then $\mathrm{d}f = \rho\,\sigma_f$ is integrable and f is Sobolev. The other direction follows directly from the proof of part a). ■

We close this section with three results on the variation of *globally integrable* functions that all additionally require *geodesic completeness*. Firstly, in the latter situation, the variation can be approximated simultaniously to the L^1-norm by the corresponding data of smooth compactly supported functions:

Theorem 2.12. *If M is geodesically complete, then for any $f \in \mathsf{L}^1(M)$ there is a sequence $(f_n) \subset \mathsf{C}_0^\infty(M)$ such that $f_n \to f$ in $\mathsf{L}^1(M)$ and $\mathrm{Var}(f_n) \to \mathrm{Var}(f)$ as $n \to \infty$.*

Proof. If $\mathrm{Var}(f) = \infty$, then any sequence $(f_n) \subset \mathsf{C}_0^\infty(M)$ such that $f_n \to f$ in $\mathsf{L}^1(M)$ satisfies $\mathrm{Var}(f_n) \to \infty$ in view of Proposition 2.8.
For the case $\mathrm{Var}(f) < \infty$, let us remark that the statement is proved in [28, Proposition 1.4] for $\mathsf{BV}_{\mathbb{R}}(M)$, the real-valued elements of $\mathsf{BV}(M)$. However, one can use the same localization argument in our complex-valued situation to reduce the assertion to domains in $\mathbb{R}^m$ (the geodesic completeness is used precisely in this highly nonstandard localization argument). In the latter case, the assertion follows from combining our Corollary 2.6 above with suitable known approximation results, which are available for $\mathbb{R}^2$-valued BV functions in the Euclidean setting (cf. Theorem 3.9 in [2]) or for real-valued BV functions with respect to *weighted* variation (cf. Theorem 3.4 in [5]). Indeed, any of the latter two results can be easily generalized to cover the vector-valued weighted case. ■

Note that Theorem 2.12 does not imply that $\mathsf{C}_0^\infty(M)$ is dense in $\mathsf{BV}(M)$.

We directly get the following corollary from combining the lower semicontinuity of the variation with Proposition 2.3, which seemingly cannot be deduced in an elementary way, that is, without the above approximation result:

Corollary 2.13. *Let g denote the underlying Riemannian structure on M and let (M, g) be geodesically complete. If g' is another Riemannian structure on M which is quasi-isometric to g, that is, if there are $C_1, C_2 > 0$ such that for all $x \in M$ one has $C_1 g_x \leq g'_x \leq C_2 g_x$ as norms[4], then the corresponding norms $\|\bullet\|_{\mathsf{BV}}$ and $\|\bullet\|'_{\mathsf{BV}}$ are equivalent.*

Finally, we note that geodesic completeness and global integrability admit an enlargement of the admissible class of test-1-forms, a result that we will also use in the proof of our main result. To this end, let

$$\Omega^1_{\mathrm{bd}}(M) := \left\{\alpha \,\middle|\, \alpha \in \Omega^1_{\mathsf{C}^\infty \cap \mathsf{L}^\infty}(M), \mathrm{d}^\dagger \alpha \in \mathsf{L}^\infty(M)\right\}.$$

[4]Note here that quasi-isometric Riemannian structures produce equivalent L^p-norms.

Now the following fact can be easily deduced from the existence of first order cut-off functions:

Lemma 2.14. *If M is geodesically complete, then for any $f \in \mathsf{L}^1(M)$ one has*

$$\mathrm{Var}(f) = \sup\left\{ \left| \int_M \overline{f(x)} \mathrm{d}^\dagger \alpha(x) \mathrm{vol}(\mathrm{d}x) \right| \,\middle|\, \alpha \in \Omega^1_{\mathrm{bd}}(M), \|\alpha\|_\infty \leq 1 \right\}.$$

Proof. The proof is essentially the same as the proof of Lemma 3.1 in [10], which considers the real-valued situation. We repeat the simple argument for the convenience of the reader.
The inequality $\mathrm{Var}(f) \leq \dots$ is trivial. For $\mathrm{Var}(f) \geq \dots$, we take a sequence of first order cut-off functions (ψ_n) as in the proof of Proposition 2.10 and let $\alpha \in \Omega^1_{\mathrm{bd}}(M)$ be such that $\|\alpha\|_\infty \leq 1$. Then one has

$$\mathrm{d}^\dagger(\psi_n \alpha) = \psi_n \mathrm{d}^\dagger \alpha - \alpha(X^{\mathrm{d}\psi_n}),$$

where $X^{\mathrm{d}\psi_n}$ is the smooth vector field on M corresponding to $\mathrm{d}\psi_n$ via the Riemannian structure, and one gets

$$\left| \int_M \overline{f(x)} \mathrm{d}^\dagger \alpha(x) \mathrm{vol}(\mathrm{d}x) \right| = \lim_{n \to \infty} \left| \int_M \overline{f(x)} \mathrm{d}^\dagger (\psi_n \alpha)(x) \mathrm{vol}(\mathrm{d}x) \right| \leq \mathrm{Var}(f),$$

where the equality follows from dominated convergence. ■

3 The heat semigroup characterization of the variation

We now come to the formulation and the proof of the main result of this note: A heat semigroup characterization of the variation of L^1- functions to a class of Riemannian manifolds with possibly unbounded from below Ricci curvature. To be precise, we will allow certain negative parts of the Ricci cuvature to be in the Kato class of M. To this end we first recall the definition of the minimal positive heat kernel $p(t, x, y)$ on M: Namely, $p(t, x, y)$ can be defined [17] as the pointwise minimal function

$$p(\bullet, \bullet, \bullet) : (0, \infty) \times M \times M \longrightarrow (0, \infty)$$

with the property that for all fixed $y \in M$, the function $p(\bullet, \bullet, y)$ is a classic $(= \mathsf{C}^{1,2})$ solution of

$$\partial_t u(t, x) = \frac{1}{2} \Delta u(t, x), \ \lim_{t \to 0+} u(t, \bullet) = \delta_y.$$

It follows from parabolic regularity that $p(t,x,y)$ is smooth in (t,x,y), and furthermore $p(t,\bullet,\bullet)$ is the unique continuous version of the integral kernel of e^{-tH}. The reader should notice that the strict positivity of $p(t,x,y)$ follows from the connectedness of M. Now we can define:

Definition 3.1. A Borel function $w: M \to \mathbb{C}$ is said to be in the *Kato class* $\mathcal{K}(M)$ of M, if

$$\lim_{t\to 0+} \sup_{x\in M} \int_0^t \int_M p(s,x,y)|w(y)|\mathrm{vol}(\mathrm{d}y)\mathrm{d}s = 0. \tag{10}$$

It is easily seen [18] that one always has the inclusions

$$\mathsf{L}^\infty(M) \subset \mathcal{K}(M) \subset \mathsf{L}^1_{\mathrm{loc}}(M),$$

but in typical applications one can say much more. To make the latter statement precise, for any $p \in [1,\infty)$ let $\mathsf{L}^p_{\mathrm{u,loc}}(M)$ denote the space of uniformly locally p-integrable functions on M, that is, a Borel function $w: M \to \mathbb{C}$ is in $\mathsf{L}^p_{\mathrm{u,loc}}(M)$, if and only if

$$\sup_{x\in M} \int_{\mathrm{K}_1(x)} |w(y)|^p \,\mathrm{vol}(\mathrm{d}y) < \infty. \tag{11}$$

Note the simple inclusions

$$\mathsf{L}^p(M) \subset \mathsf{L}^p_{\mathrm{u,loc}}(M) \subset \mathsf{L}^p_{\mathrm{loc}}(M).$$

Now one has the following result, which essentially states that a Gaussian upper bound for $p(t,x,y)$ implies $\mathsf{L}^p(M) \subset \mathcal{K}(M)$ for suitable $p = p(m)$, and that with a little more control on the Riemannian structure one even has $\mathsf{L}^p_{\mathrm{u,loc}}(M) \subset \mathcal{K}(M)$ (cf. Proposition 2.4 in [19]):

Proposition 3.2. *Let p be such that $p \geq 1$ if $m = 1$, and $p > m/2$ if $m \geq 2$.*
a) *If there is $C > 0$ and a $t_0 > 0$ such that for all $0 < t \leq t_0$ and all $x \in M$ one has $p(t,x,x) \leq Ct^{-\frac{m}{2}}$, then one has*

$$\mathsf{L}^p(M) + \mathsf{L}^\infty(M) \subset \mathcal{K}(M). \tag{12}$$

b) *Let M be geodesically complete, and assume that there are constants $C_1,\dots,C_6,t_0 > 0$ such that for all $0 < t \leq t_0$, $x,y \in M$, $r > 0$ one has* $\mathrm{vol}(\mathrm{K}_r(x)) \leq C_1 r^m \mathrm{e}^{C_2 r}$ *and*

$$C_3 t^{-\frac{m}{2}} \mathrm{e}^{-C_4 \frac{\mathrm{d}(x,y)^2}{t}} \leq p(t,x,y) \leq C_5 t^{-\frac{m}{2}} \mathrm{e}^{-C_6 \frac{\mathrm{d}(x,y)^2}{t}}.$$

Then one has

$$\mathsf{L}^p_{\mathrm{u,loc}}(M) + \mathsf{L}^\infty(M) \subset \mathcal{K}(M). \tag{13}$$

We refer the reader to [18] and particularly to [25] for several further global and local aspects on $\mathcal{K}(M)$.
In the sequel, we will consider the Ricci curvature $\mathscr{R}$ of M as a smooth, self-adjoint section in the smooth complex vector bundle $\mathrm{End}\,(\mathrm{T}^*M) \to M$, whose quadratic form is defined pointwise through the trace of the Riemannian curvature tensor of M.
With these notions at hand, the following dynamical characterization of the variation of globally integrable functions is the main result of this note:

Theorem 3.3. *Let M be geodesically complete and assume that $\mathscr{R}$ admits a decomposition $\mathscr{R} = \mathscr{R}_1 - \mathscr{R}_2$ into pointwise self-adjoint Borel sections $\mathscr{R}_1, \mathscr{R}_2 \geq 0$ in $\mathrm{End}(\mathrm{T}^*M)$ such that $|\mathscr{R}_2| \in \mathcal{K}(M)$. Then for any $f \in \mathsf{L}^1(M)$ one has*

$$\mathrm{Var}(f) = \lim_{t \to 0+} \int_M \left| \mathrm{d}\mathrm{e}^{-tH} f(x) \right|_x \mathrm{vol}(\mathrm{d}x). \tag{14}$$

Remark 3.4. If M is the Euclidean $\mathbb{R}^m$, then (14) has been proven (for real-valued $f's$) by De Giorgi [12] in 1954. De Giorgi's result has been extended to geodesically complete Riemannian manifolds first by Miranda/ the second author/Paronetto/Preunkert [28] in 2007, under the assumptions that M has Ricci curvature $\mathscr{R}$ bounded below and satisfies the nontrapping condition

$$\inf_{x \in M} \mathrm{vol}(\mathrm{K}_1(x)) > 0. \tag{15}$$

Again in 2007, Carbonaro-Mauceri [10] have removed condition (15), giving a much simpler proof that relies on an L^∞-estimate for e^{-tH_1} due to Bakry [6].
The point we want to make here is that a large part of the technique from [10] is flexible enough to deal with certain unbounded "negative parts" of $\mathscr{R}$. The essential observation is that in view of Weitzenböck's formula for $-\Delta_1$, e^{-tH_1} becomes a generalized Schrödinger semigroup with potential $\mathscr{R}/2$, which, under our assumptions on M, is given by a Feynman-Kac type path integral formula. This follows from the abstract work of the first author on generalized Schrödinger semigroups [20]. Through semigroup domination, the latter formula makes it possible to prove (see Lemma 3.9 below) a bound of the form

$$\left\| \mathrm{e}^{-tH_1} \mid_{\Omega^1_{\mathsf{L}^2 \cap \mathsf{L}^\infty}(M)} \right\|_\infty \leq \delta \mathrm{e}^{tC(\delta)} \text{ for all } t \geq 0,\, \delta > 1,$$

which is weaker for small times than the above mentioned L^∞-estimate by Bakry for the case $\mathscr{R} \geq -C$ (the latter is the form $\cdots \leq \mathrm{e}^{tC}$), but turns out to be just enough to extend a large part of the methods of [10] to our more

general setting. Finally, let us also point out that heat semigroup characterizations of BV have also been derived in other situations than functions on Riemannian manifolds (cf. [16] [3] [9]). We propose two further extensions of the setting of Theorem 3.3:

- Definition 2.2 suggests that one can define the notion of a "D-variation" for sections in vector bundles, instead of functions, where "d" has to be replaced by an appropriate first order linear differential operator D acting between sections. Here, in principle, the results from [15] on path integral formulae for the derivatives of geometric Schrödinger semigroups could be very useful.

- As all of the data in (14) have analogues (see for example [14, 27]) on discrete metric graphs, it would certainly be also interesting to see to what extent such a result can be proved in the infinite discrete setting.

Before we come to the proof of Theorem 3.3, we continue with several consequences of the latter result. Firstly, in view of Proposition 3.2, we directly get the following criterion:

Corollary 3.5. a) *Under the assumptions of Proposition 3.2 a), assume that $\mathscr{R}$ admits a decomposition $\mathscr{R} = \mathscr{R}_1 - \mathscr{R}_2$ into pointwise self-adjoint Borel sections $\mathscr{R}_1, \mathscr{R}_2 \geq 0$ in $\mathrm{End}(\mathrm{T}^*M)$ such that $|\mathscr{R}_2| \in \mathsf{L}^p(M) + \mathsf{L}^\infty(M)$. Then one has (14).*
b) *Under the assumptions of Proposition 3.2 b), assume that $\mathscr{R}$ admits a decomposition $\mathscr{R} = \mathscr{R}_1 - \mathscr{R}_2$ into pointwise self-adjoint Borel sections $\mathscr{R}_1, \mathscr{R}_2 \geq 0$ in $\mathrm{End}(\mathrm{T}^*M)$ such that $|\mathscr{R}_2| \in \mathsf{L}^p_{\mathrm{u,loc}}(M) + \mathsf{L}^\infty(M)$. Then one has (14).*

We remark that a somewhat comparable assumption on the Ricci curvature as in part a) of the above corollary has also been made in Theorem 2.1 in [11], where the authors prove certain large time bounds on the integral kernel of e^{-tH_1}.

Theorem 3.3 makes it also possible to derive a stability of (14) under certain conformal transformations which is particularly useful in the Euclidean $\mathbb{R}^m$.

Remark 3.6. 1. If g denotes the fixed Riemannian structure on M and if $\psi \in \mathsf{C}^\infty_{\mathbb{R}}(M)$, then we can define a new Riemannian structure on M by setting $g_\psi := \mathrm{e}^{2\psi} g$. Cearly, a section in $\mathrm{End}(\mathrm{T}^*M)$ is self-adjoint with respect to g, if and only if it is self-adjoint with respect to g_ψ.

2. The Riemannian structures g and g_ψ are quasi-isometric if ψ is bounded, so that then $\mathsf{L}^p(M; g_\psi) = \mathsf{L}^p(M)$, as well as $\mathsf{L}^p_{\mathrm{u,loc}}(M; g_\psi) = \mathsf{L}^p_{\mathrm{u,loc}}(M)$ for all

p. Clearly, the boundedness of ψ also implies that a self-adjoint section in $\mathrm{End}(\mathrm{T}^*M)$ is bounded from below with respect to g, if and only if it is so with respect to g_ψ.

3. If we denote with $\mathscr{R}_\psi$ the Ricci curvature with respect to g_ψ, then one has the perturbation formula (see for example Theorem 1.159 in [7]) $\mathscr{R}_\psi = \mathscr{R} + \mathscr{T}_\psi$, where $\mathscr{T}_\psi$ is the smooth self-adjoint section in $\mathrm{End}(\mathrm{T}^*M)$ given by

$$\mathscr{T}_\psi := (2-m)\Big(\mathrm{Hess}(\psi) - \mathrm{d}\psi \otimes \mathrm{d}\psi\Big) - \Big(\Delta\psi + (m-2)|\mathrm{d}\psi|^2\Big)g. \tag{16}$$

It is clear from (16) that $\mathscr{R}_\psi$ need not be bounded from below, even if $\mathscr{R}$ is bounded from below and ψ is bounded.

Now we can prove the following result which uses the machinery of parabolic Harnack inequalities:

Corollary 3.7. *Let $\psi \in \mathsf{C}^\infty_{\mathbb{R}}(M)$ be bounded, let M be geodesically complete and let p be such that $p \geq 1$ if $m = 1$, and $p > m/2$ if $m \geq 2$. Furthermore, assume that there are $C_1, C_2, R > 0$ with the following property: one has $\mathscr{R} \geq -C_1$ and*

$$\mathrm{vol}(\mathrm{K}_r(x)) \geq C_2 r^m \quad \text{for all } 0 < r \leq R,\ x \in M. \tag{17}$$

If $\mathscr{T}_\psi$ admits a decomposition $\mathscr{T}_\psi = \mathscr{T}_1 - \mathscr{T}_2$ into pointwise self-adjoint Borel sections $\mathscr{T}_1, \mathscr{T}_2 \geq 0$ in $\mathrm{End}(\mathrm{T}^*M)$ such that

$$|\mathscr{T}_2| \in \mathsf{L}^p_{\mathrm{u,loc}}(M; g_\psi) + \mathsf{L}^\infty(M; g_\psi),$$

then for any $f \in \mathsf{L}^1(M; g_\psi)$ one has (14) with respect to g_ψ.

Proof. Firstly, we note the classical fact that $\mathscr{R} \geq -C_1$ implies Li-Yau's estimate for all $t > 0, x, y \in M$,

$$\frac{C_3}{\mathrm{vol}(\mathrm{K}_{\sqrt{t}}(x))}\mathrm{e}^{-C_4\frac{\mathrm{d}(x,y)^2}{t}} \leq p(t,x,y) \leq \frac{C_5}{\mathrm{vol}(\mathrm{K}_{\sqrt{t}}(x))}\mathrm{e}^{-C_6\frac{\mathrm{d}(x,y)^2}{t}}, \tag{18}$$

thus using

$$\mathrm{vol}(\mathrm{K}_r(x)) \leq C_7 r^m \mathrm{e}^{C_8 r} \text{ for all } r > 0, \tag{19}$$

which is a simple consequence of Bishops's volume comparison theorem (see for example p. 7 in [22]), we can deduce the inequality

$$C_9 t^{-\frac{m}{2}}\mathrm{e}^{-C_{10}\frac{\mathrm{d}(x,y)^2}{t}} \leq p(t,x,y) \leq C_{11} t^{-\frac{m}{2}}\mathrm{e}^{-C_{12}\frac{\mathrm{d}(x,y)^2}{t}} \quad \text{for all } 0 < t \leq 1. \tag{20}$$

By Theorem 5.5.3 in [30], Li-Yau's inequality is equivalent to the conjunction of the local Poincaré inequality and volume doubling, which by Theorem 5.5.1 in [30] is equivalent to the validity of the parabolic Harnack inequality. The latter inequality is stable under a change to an quasi-isometric Riemannian structure by Theorem 5.5.9 in [30], so that we also have (18) with respect to g_ψ. Again using the quasi-isometry of the Riemannian structures, it is clear that we also have (17) and (19), and thus (20) with respect to g_ψ. But now we can use Corollary 3.5 to deduce (14) with respect to g_ψ, keeping in mind that the negative part $\mathscr{R}_-$ is bounded by assumption. ■

The rest of this paper is devoted to the proof of Theorem 3.3, which will require two more auxiliary results. To this end, we have to introduce some probabilistic notation first.

Let $(\Omega, \mathscr{F}, \mathscr{F}_*, \mathbb{P})$ be a filtered probability space which satisfies the usual assumptions. We assume that $(\Omega, \mathscr{F}, \mathscr{F}_*, \mathbb{P})$ is chosen in a way such that it carries an appropriate family of Brownian motions $B(x)\colon [0, \zeta(x)) \times \Omega \to M$, $x \in M$, where $\zeta(x) : \Omega \to [0, \infty]$ is the lifetime of $B(x)$. The well-known relation [23]

$$\mathbb{P}\{B_t(x) \in N, t < \zeta(x)\} = \int_N p(t,x,y)\mathrm{vol}(\mathrm{d}y) \quad \text{for any Borel set } N \subset M$$

implies directly that for a Borel function $w : M \to \mathbb{C}$ one has $w \in \mathcal{K}(M)$, if and only if

$$\lim_{t \to 0+} \sup_{x \in M} \mathbb{E}\left[\int_0^t |w(B_s(x))|\, 1_{\{s<\zeta(x)\}} \mathrm{d}s\right] = 0,$$

which is the direct link between Theorem 3.3 and probability theory. We will need the following subtle generalization of Proposition 2.5 from [20], which does not require any control on the Riemannian structure of M:

Lemma 3.8. *For any $v \in \mathcal{K}(M)$ and any $\delta > 1$ there is a $C(v,\delta) > 0$ such that for all $t \geq 0$,*

$$\sup_{x \in M} \mathbb{E}\left[\mathrm{e}^{\int_0^t |v(B_s(x))| \mathrm{d}s} 1_{\{t<\zeta(x)\}}\right] \leq \delta \mathrm{e}^{tC(v,\delta)}. \tag{21}$$

Proof. The proof is an adaption of that of Proposition 2.5 from [20] (see particularly also [13], [35], [1]). We give a detailed proof here for the convenience of the reader. Let us first state two abstract facts:

1. With $\hat{M} = M \cup \{\infty_M\}$ the Alexandroff compactification of M, we can canonically extend any Borel function $w : M \to \mathbb{C}$ to a Borel function $\hat{w}$:

$M \to \mathbb{C}$ by setting $\hat{w}(\infty_M) = 0$, and $B(x)$ to a process $\hat{B}(x) : [0,\infty)\times\Omega \to \hat{M}$ by setting $\hat{B}_s(x)(\omega) := \infty_M$, if $s \geq \zeta(x)(\omega)$. Then one trivially has

$$\mathbb{E}\left[\mathrm{e}^{\int_0^t |w(B_s(x))| \mathrm{d}s} 1_{\{t<\zeta(x)\}}\right] \leq \mathbb{E}\left[\mathrm{e}^{\int_0^t \left|\hat{w}(\hat{B}_s(x))\right| \mathrm{d}s}\right]. \tag{22}$$

2. For any Borel function $w : M \to \mathbb{C}$ and any $s \geq 0$ let

$$\begin{aligned} D(w,s) &:= \sup_{x\in M} \mathbb{E}\left[\int_0^s \left|\hat{w}(\hat{B}_r(x))\right| \mathrm{d}r\right] \\ &= \sup_{x\in M} \mathbb{E}\left[\int_0^t |w(B_s(x))|\, 1_{\{s<\zeta(x)\}} \mathrm{d}s\right] \in [0,\infty], \end{aligned}$$

and

$$\tilde{D}(w,s) := \sup_{x\in M} \mathbb{E}\left[\mathrm{e}^{\int_0^s \left|\hat{w}(\hat{B}_r(x))\right| \mathrm{d}r}\right] \in [0,\infty].$$

Then *Kas'minskii's Lemma* states that the following assertion holds:

$$\text{For any } s > 0 \text{ with } D(w,s) < 1 \text{ one has } \tilde{D}(w,s) \leq \frac{1}{1 - D(w,s).} \tag{23}$$

This estimate can be proved as follows: For any $n \in \mathbb{N}$ let

$$s\sigma_n := \left\{ q = (q_1,\dots,q_n) \,\middle|\, 0 \leq q_1 \leq \dots \leq q_n \leq s \right\} \subset \mathbb{R}^n$$

denote the s-scaled standard simplex. Then it is sufficient to prove that for all n one has

$$\begin{aligned} \tilde{D}_n(w,s) &:= \sup_{x\in M} \int_{s\sigma_n} \mathbb{E}\left[\left|\hat{w}(\hat{B}_{q_1}(x))\right| \dots \left|\hat{w}(\hat{B}_{q_n}(x))\right|\right] \mathrm{d}^n q \\ &\leq \frac{1}{1 - D(w,s)} \tilde{D}_{n-1}(w,s). \end{aligned} \tag{24}$$

But the Markoff property of $B(x)$ implies

$$\begin{aligned} \tilde{D}_n(w,s) = \sup_{x\in M} \int_{s\sigma_{n-1}} & \mathbb{E}\left[\left|\hat{w}(\hat{B}_{q_1}(x))\right| \dots \left|\hat{w}(\hat{B}_{q_{n-1}}(x))\right| \times \right. \\ & \left. \times \mathbb{E}\left[\int_0^{s-q_{n-1}} \left|\hat{w}(\hat{B}_u(y))\right| \mathrm{d}u\right] |_{y=\hat{B}_{q_{n-1}}(x)}\right] \mathrm{d}^{n-1} q \\ \leq \frac{1}{1 - D(w,s)} \tilde{D}_{n-1}(w,s), & \end{aligned} \tag{25}$$

which proves Kas'minskii's lemma.

Using the two observations above, the actual proof of (21) can be carried out as follows: By the Kato property of v, we can pick an $s(v,\delta) > 0$ with $D(v, s(v,\delta)) < 1 - 1/\delta$. Let $n \in \mathbb{N}$ be large enough with $t \leq (n+1)s(v,\delta)$. Then the Markoff property of $B(x)$ and Kas'minskii's Lemma imply

$$
\begin{aligned}
&\tilde{D}(v,t)\\
&\leq \tilde{D}(v,(n+1)s(v,\delta))\\
&= \sup_{x\in M} \mathbb{E}\left[\mathrm{e}^{\int_0^{ns(v,\delta)}\left|\hat{v}(\hat{B}_r(x))\right|\mathrm{d}r}\mathbb{E}\left[\mathrm{e}^{\int_0^{s(v,\delta)}\left|\hat{v}(\hat{B}_r(y))\right|\mathrm{d}r}\right]\Big|_{y=\hat{B}_{ns(v,\delta)}(x)}\right]\\
&\leq \frac{1}{1-D(v,s(v,\delta))}\tilde{D}(v,ns(v,\delta))\\
&= \frac{1}{1-D(v,s(v,\delta))}\times\\
&\quad\times \sup_{x\in M} \mathbb{E}\left[\mathrm{e}^{\int_0^{(n-1)s(v,\delta)}\left|\hat{v}(\hat{B}_r(x))\right|\mathrm{d}r}\mathbb{E}\left[\mathrm{e}^{\int_0^{s(v,\delta)}\left|\hat{v}(\hat{B}_r(y))\right|\mathrm{d}r}\right]\Big|_{y=\hat{B}_{(n-1)s(v,\delta)}(x)}\right]\\
&\leq \ldots\ (n\text{-times})\\
&\leq \frac{1}{1-D(v,s(v,\delta))}\left(\frac{1}{1-D(v,s(v,\delta))}\right)^n\\
&\leq \frac{1}{1-D(v,s(v,\delta))}\mathrm{e}^{\frac{t}{s(v,\delta)}\log\left(\frac{1}{1-D(v,s(v,\delta))}\right)}\\
&< \delta\ \mathrm{e}^{\frac{t}{s(v,\delta)}\log\left(\frac{1}{1-D(v,s(v,\delta))}\right)},
\end{aligned}
$$

which proves (21) in view of (22). ■

The latter result will be used to deduce:

Lemma 3.9. *Under the assumptions of Theorem 3.3, for any $\delta > 1$ there is a $C(\delta) > 0$ such that for any $t \geq 0$ and any $\alpha \in \Omega^1_{\mathsf{L}^2\cap\mathsf{L}^\infty}(M)$ one has*

$$
\left\|\mathrm{e}^{-tH_1}\alpha\right\|_\infty \leq \delta \mathrm{e}^{tC(\delta)}\left\|\alpha\right\|_\infty. \tag{26}
$$

Proof. The Weitzenböck formula states that $-\Delta_1/2 = \nabla_1^\dagger\nabla_1/2 + \mathscr{R}/2$, where ∇_1 stands for the Levi-Civita connection acting on 1-forms. Under the given assumptions on $\mathscr{R}$, we can use Theorem 2.13 in [18] to define the form sum $\tilde{H}_1$ of the Friedrichs realization of $\nabla_1^\dagger\nabla_1/2$ and the multiplication operator $\mathscr{R}/2$ in $\Omega^k_{\mathsf{L}^2}(M)$. But the geodesic completeness assumption implies the essential self-adjointness of $-\Delta_1$ on the domain of definition $\Omega^1_{\mathsf{C}^\infty_0}(M)$ (this essential self-adjointness is a classical result [33]; under our assumption on $\mathscr{R}$, this also follows from the main result of [19]), and as a consequence we get $H_1 = \tilde{H}_1$. We define scalar potentials $w_j : M \to [0,\infty)$, $w : M \to \mathbb{R}$ by

$$
w_1 := \min\sigma(\mathscr{R}_1/2(\bullet)),\ w_2 := \max\sigma(\mathscr{R}_2/2(\bullet)),\ w := w_1 - w_2,
$$

where $\sigma(\mathscr{R}_j/2(x))$ stands for the spectrum of the nonnegative self-adjoint operator $\mathscr{R}_j/2(x) : \mathrm{T}_x^*M \to \mathrm{T}_x^*M$. Then clearly $w_1 \in \mathsf{L}^1_{\mathrm{loc}}(M)$, $w_2 \in \mathcal{K}(M)$, and the above considerations combined with Theorem 2.13 from [20] (semigroup domination), Theorem 2.9 from [20] (a scalar Feynman-Kac formula) and $-w \leq w_2$ imply the first inequality in

$$\begin{aligned} \left|\mathrm{e}^{-tH_1}\alpha(x)\right|_x &\leq \mathbb{E}\left[\mathrm{e}^{\int_0^t w_2(B_s(x))\mathrm{d}s}|\alpha|(B_t(x))1_{\{t<\zeta(x)\}}\right] \\ &\leq \|\alpha\|_\infty \,\mathbb{E}\left[\mathrm{e}^{\int_0^t w_2(B_s(x))\mathrm{d}s}1_{\{t<\zeta(x)\}}\right] \ \text{ for a.e. } x \in M. \end{aligned} \tag{27}$$

But now the assertion follows readily from Lemma 3.8. ∎

Now we can prove Theorem 3.3:

Proof of Theorem 3.3. Firstly, the inequality

$$\mathrm{Var}(f) \leq \liminf_{t\to 0+} \int_M \left|\mathrm{d}\mathrm{e}^{-tH}f(x)\right|_x \mathrm{vol}(\mathrm{d}x)$$

can be deduced exactly as in the proof of Theorem 3.2 in [10]. In fact, this inequality is always satisfied without any assumptions on the Riemannian structure of M. To see this, just note that if $\alpha \in \Omega^1_{\mathsf{C}_0^\infty}(M)$ is such that $\|\alpha\|_\infty \leq 1$, then

$$\begin{aligned} \left|\int_M \overline{f(x)}\mathrm{d}^\dagger\alpha(x)\mathrm{vol}(\mathrm{d}x)\right| &= \left|\lim_{t\to 0+}\int_M \overline{\mathrm{e}^{-tH}f(x)}\mathrm{d}^\dagger\alpha(x)\mathrm{vol}(\mathrm{d}x)\right| \\ &= \left|\lim_{t\to 0+}\int_M (\mathrm{d}\mathrm{e}^{-tH}f(x), \alpha(x))_x\mathrm{vol}(\mathrm{d}x)\right| \\ &\leq \liminf_{t\to 0+}\int_M \left|\mathrm{d}\mathrm{e}^{-tH}f(x)\right|_x \mathrm{vol}(\mathrm{d}x), \end{aligned} \tag{28}$$

where we have used $\left\|\mathrm{e}^{-tH}f - f\right\|_1 \to 0$ as $t \to 0+$.
In order to prove

$$\mathrm{Var}(f) \geq \limsup_{t\to 0+} \int_M \left|\mathrm{d}\mathrm{e}^{-tH}f(x)\right|_x \mathrm{vol}(\mathrm{d}x), \tag{29}$$

we first remark the well-known fact (see for example the appendix of [15]) that geodesic completeness implies

$$\mathrm{e}^{-tH_1}\mathrm{d}h = \mathrm{d}\mathrm{e}^{-tH}h \ \text{ for any } h \in \mathsf{C}_0^\infty(M). \tag{30}$$

Now let $\alpha \in \Omega^1_{\mathsf{C}_0^\infty}(M)$ be such that $\|\alpha\|_\infty \leq 1$, and let $t, \epsilon > 0$ be arbitrary. Then Lemma 3.9 shows

$$\left\|\mathrm{e}^{-tH_1}\alpha\right\|_\infty \leq (1+\epsilon)\mathrm{e}^{tC(\epsilon)}, \tag{31}$$

and applying (30) with $h = \mathrm{d}^\dagger\alpha$ gives (by testing against $\tilde{h} \in \mathsf{C}_0^\infty(M)$) the identity $\mathrm{d}^\dagger \mathrm{e}^{-tH_1}\alpha = \mathrm{e}^{-tH}\mathrm{d}^\dagger\alpha$. So using $\mathrm{e}^{-tH} \in \mathscr{L}(\mathsf{L}^\infty(M))$, we can conclude $\mathrm{e}^{-tH_1}\alpha \in \Omega^1_{\mathrm{bd}}(M)$. Finally, combining $\mathrm{e}^{-tH_1}\alpha \in \Omega^1_{\mathrm{bd}}(M)$, (31), Lemma 2.14 and $\mathrm{d}^\dagger \mathrm{e}^{-tH_1}\alpha = \mathrm{e}^{-tH}\mathrm{d}^\dagger\alpha$, we have

$$
\begin{aligned}
\left| \int_M \overline{\mathrm{e}^{-tH} f(x)} \mathrm{d}^\dagger \alpha(x) \mathrm{vol}(\mathrm{d}x) \right| &= \left| \int_M \overline{f(x)} \mathrm{d}^\dagger \mathrm{e}^{-tH_1} \alpha(x) \mathrm{vol}(\mathrm{d}x) \right| \\
&\leq (1+\epsilon)\mathrm{e}^{tC(\epsilon)} \mathrm{Var}(f), \qquad (32)
\end{aligned}
$$

so that taking the supremum over all such α and using Proposition 2.3 we arrive at

$$
\begin{aligned}
\int_M \left| \mathrm{d}\mathrm{e}^{-tH} f(x) \right|_x \mathrm{vol}(\mathrm{d}x) &= \sup_\alpha \left| \int_M \overline{\mathrm{e}^{-tH} f(x)} \mathrm{d}^\dagger \alpha(x) \mathrm{vol}(\mathrm{d}x) \right| \\
&\leq (1+\epsilon)\mathrm{e}^{tC(\epsilon)} \mathrm{Var}(f), \qquad (33)
\end{aligned}
$$

which proves (29) by first taking $\limsup_{t\to 0+}$, and then taking $\lim_{\epsilon \to 0+}$. ■

A Vector measures on locally compact spaces

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$, and denote the corresponding standard inner-product and norm on $\mathbb{K}^m$ with $(\bullet,\bullet)_{\mathbb{K}^m}$ and $|\bullet|_{\mathbb{K}^m}$. Let X be a locally compact Hausdorff space with its Borel-σ-algebra $\mathcal{B}(X)$. We denote with $\mathsf{C}_\infty(X,\mathbb{K}^m)$ the space of $\mathbb{K}^m$-valued functions on X that vanish at infinity, which is a $\mathbb{K}$-Banach space with respect to the uniform norm $\|\bullet\|_\infty$.
A $\mathbb{K}^m$*-valued Borel measure* on X is defined to be a countably additive set function $\nu : \mathcal{B}(X) \to \mathbb{K}^m$, and its *total variation* measure is the positive finite Borel measure defined for $B \in \mathcal{B}(X)$ by

$$
|\nu|(B) = \sup \left\{ \sum_{j=1}^\infty |\nu(B_j)|_{\mathbb{K}^m} \,\middle|\, B_j \in \mathcal{B}(X) \text{ for all } j \in \mathbb{N}, B = \bigsqcup_{j=1}^\infty B_j \right\}.
$$

Of course one has $\nu = |\nu|$, in case ν itself is a positive finite Borel measure. Let us denote the $\mathbb{K}$-linear space of $\mathbb{K}^m$-valued Borel measure on X with $\tilde{\mathscr{M}}(X,\mathbb{K}^m)$. Then one has:

Proposition A.1. a) *$\tilde{\mathscr{M}}(X,\mathbb{K}^m)$ is a $\mathbb{K}$-Banach space with respect to the total variation norm, and the map*

$$
\tilde{\Psi} : \tilde{\mathscr{M}}(X,\mathbb{K}^m) \longrightarrow (\mathsf{C}_\infty(X,\mathbb{K}^m))^*, \;\; \tilde{\Psi}(\nu)(f) = \int_X (f, \mathrm{d}\nu)_{\mathbb{K}^m}
$$

is an isometric isomorphism of $\mathbb{K}$-linear spaces. If $m = 1$, then $\tilde{\Psi}$ is order preserving.
b) *The map*

$$\tilde{\mathscr{M}}(X, \mathbb{C}^m) \longrightarrow \tilde{\mathscr{M}}(X, \mathbb{R}^{2m}), \ \ \nu \longmapsto \nu_{\mathbb{R}^{2m}} := (\mathrm{Re}(\nu), \mathrm{Im}(\nu))$$

is an isometric isomorphism of $\mathbb{R}$-linear spaces.

Proof. Part a) is just a variant of Riesz-Markoff's theorem, see e.g. [29, Theorem 6.19]. In part b) it is clear that the indicated map is an $\mathbb{R}$-linear isomorphism. The essential observation for the proof of $|\nu| = |\nu_{\mathbb{R}^{2m}}|$ is the polar decomposition: Namely, by a Radon-Nikodym-type argument one gets the existence of a Borel function $\sigma : X \to \mathbb{C}^m$, with $|\sigma| = 1$ $|\nu|$-a.e. in X, such that $\mathrm{d}\nu = \sigma \ \mathrm{d}|\nu|$. As the norms $|\bullet|_{\mathbb{R}^{2m}}$ and $|\bullet|_{\mathbb{C}^m}$ are equal under the canonical identification of $\mathbb{R}^{2m}$ with $\mathbb{C}^m$, it follows directly from the definition of the total variation measure that $|\nu| = |\nu_{\mathbb{R}^{2m}}|$. ■

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